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Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. EM-126

Higher Order Approximations in Multiple Scattering

I. Two-Dimensional Scalar Case

II. Three-Dimensional Scalar Case

NORMAN ZITRON and SAMUEL N. KARP

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Abstract

I. A formula is derived which expresses the perturbed scattering amplitudes of a combination of two arbitrary cylinders as a function of the unperturbed scattering amplitudes of the individual cylinders. The formula is valid when the spacing of the scatterers is large compared to their dimensions. It involves derivatives of the scattering amplitudes with respect to the angles of incidence and of obversation. Interaction terms of degrees $d^{-1/2}$, d^{-1}, and $d^{-3/2}$ are taken into account, where d is the spacing. Verification is obtained in a special case. The result is employed to calculate the total scattering cross section.

II. The method of I is extended to cover the three dimensional scalar problem for two bodies of arbitrary shape. All interaction terms of order d^{-1} and d^{-2} are given.

Table of Contents

I. Two Dimensional Case

1. Introduction

The present paper deals with the diffraction of plane electromagnetic or acoustic waves by a pair of parallel cylinders of arbitrary shape. The diffraction by each cylinder, in isolation, is assumed known, and the diffraction by the configuration is calculated explicitly, in terms of these data. An approximation is involved which will be described below.

The question of multiple scattering has already been treated, but in less detail, by a number of writers. A brief sketch of the history of the problem follows, with emphasis on those treatments of the problem whose accuracy increases with the spacing. We recall Reiche and Schafer[1], who were the first to have given a wave theoretical discussion of the finite grating of circular cylinders. These authors neglected the interaction between the cylinders; their work was therefore valid, in principle, in the limit of large spacing. A very general expression for the diffraction by an arbitrary assemblage of circular cylinders was given by Twersky [2], who took all orders of interaction into account. The method depended heavily upon the separability of the circular geometry, and the most general form of the result was too complicated to be discussed. However, Twersky found that, if he proceeded to the limit of large spacing, he could simplify his result immensely and achieve a perspicuous discussion of the correction to single scattering. This work gave a correct account of terms of degree $d^{-1/2}$ and d^{-1} in the spacing.

This success led to a general investigation of the large spacing approximation by Karp [3], who showed that for cylinders of arbitrary shape, the leading terms of the <u>interaction correction</u> could themselves

be expressed explicitly in terms of non-interaction or single scattering results, the latter being regarded as given. In fact the interaction term could be regarded as being composed of the response of each cylinder to a plane wave arriving from the direction of the other cylinders. The techniques of [4] were exploited by Karp and Radlow* in the analysis of a grating of cylinders. Similar methods were used by Karp and Russek [6] in expressing the approximate solution to the problem of diffraction by a wide slit in terms of the well known solution for the half plane problem.

The purpose of the present paper is to extend the work of Karp [3] so as to take into account higher order terms. Just as in [3], the cylinders are arbitrary, and the scattering by each cylinder in isolation is assumed as given. But, it is found** that even the higher order correction terms can be calculated generally, simply and explicitly in terms of the single scattering data used in [3] for calculation of the leading terms. This is the principal result of the present paper.

The general calculation was carried out so as to include all effects of order of magnitude $d^{-1/2}$, d^{-1} and $d^{-3/2}$, where d is the spacing of the cylinders. For purposes of comparison, Twersky's calculation for a pair of circular cylinders was continued so as to include terms of order $d^{-3/2}$; this special calculation by the repeated application of addition theorems was then shown to agree with the general result of the present work, when the latter result is specialized to the case of circular cylinders. The result is used to calculate the total cross section for a pair of circular cylinders as a function of the spacing and the known unperturbed (or non-interaction) scattering amplitude functions.

*See also [5]

**Cf. [7]

2. A Statement of the Problem, and of the Method of Analysis.

We would like to know the scattering pattern of a combination of two infinite parallel cylinders in terms of the scattering patterns which these cylinders would have if they were isolated from each other. In other words, we want to obtain a functional relationship between the unperturbed and the perturbed scattering patterns of the cylinders. Such a relation is desirable because it simplifies the calculation of the pattern for the combination. If we can calculate the unperturbed patterns, we need only insert them in this relation to obtain the perturbed patterns. The relation is useful, moreover, even if the shapes of the cylinders are so complicated that we cannot separate variables or if calculation by separation of variables is too tedious*. Also in such cases, the unperturbed patterns might be measured experimentally for all angles of observation and these unperturbed patterns might then be substituted into the relation obtained here to yield the perturbed patterns.

The situation is the following: A plane wave of unit amplitude is incident upon the two parallel cylinders A and B (Figure 1).

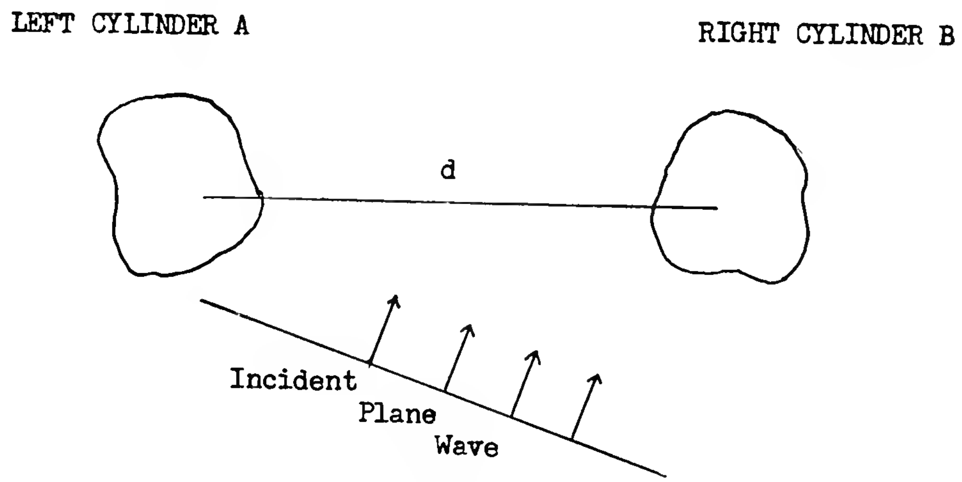

FIGURE 1

*See [4] for the use of an even less accurate relation for theoretical purposes.

To avoid ambiguity in the definition of the spacing d, we define a coordinate system for each cylinder. Let A' and B' be circular cylinders circumscribed about A and B respectively. Let a and b be the respective radii of A' and B'. We shall let Z_a be the axis of A' and Z_b be the axis of B'. The problem is two-dimensional, and we shall operate in a plane perpendicular to the Z axes. The respective intersections of this plane with the Z_a and Z_b axes will then be the origins of the coordinate systems of A and B . We can now define the spacing d as the distance between the axes of A' and B'- that is, the distance between the two origins (Figure 2).

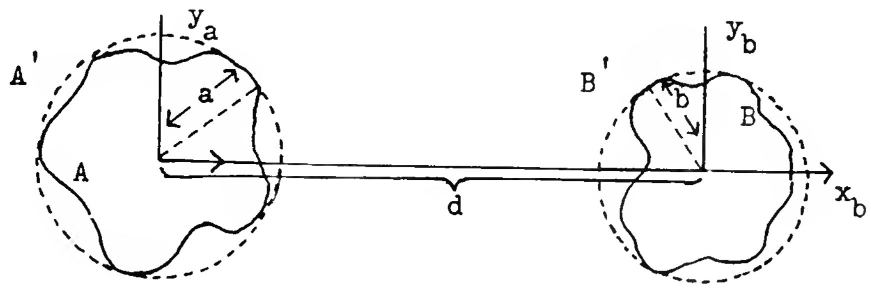

FIGURE 2

We make the following assumptions:

1) The individual complex scattering pattern is known when each cylinder stands alone in space.

2) d > > λ where λ is the wave length of the incident wave.*

3) d > > a and d > > b.

---

*Numerical calculation in [5], which are based on a less accurate procedure, showed that d could be as small as one wavelength without impairing the accuracy materially. The present result would therefore allow an even smaller spacing, provided the cylinders are kept sufficiently small compared to the spacing.

Our object is to find a functional relation between the scattering pattern of the combination and those of the isolated component cylinders. This cannot be accomplished by simple superposition of the unperturbed patterns of the components since the individual pattern of each component cylinder is modified by the field scattered by the other cylinder. We must, therefore, consider the interaction.

We shall assume that the response U_s of each cylinder to a plane wave is of the form

$$(1) \qquad U_s = \frac{e^{ikr}}{\sqrt{r}} \sum_{n=o}^{\infty} \frac{f_n(\theta,\theta_o)}{r^n} \qquad \text{for large } r$$

where r is the distance from the axis of the circumscribed circular cylinder, θ is the angle of observation, and θ_o is the angle of incidence. Both θ and θ_o are measured from the x axis of each cylinder. The x axes are collinear with d.

As will be explained below, at a sufficiently large distance from the cylinder, the scattered field U_s resembles a plane wave. This approximate plane wave elicits a response from the other cylinder, perturbing its scattered field. This response also has the form (1) and in turn perturbs the field scattered by the first cylinder. We can carry out successive calculations for this process until the desired order of accuracy is obtained. When the perturbed patterns have been calculated, they can be superposed.

We shall deal, in this paper, with interaction terms of degrees $d^{-1/2}$, d^{-1} and $d^{-3/2}$. The procedure involves a new kind of expansion of

the waves scattered by each cylinder about the origin located in the other cylinder. In order to insure that all terms up to order $d^{-3/2}$ are contained in the result, we must include them in the first expansion. We then find that the field scattered by a given cylinder can be represented, in the neighborhood of the other cylinder, as a plane wave, plus additional terms which are recognized as derivatives of a plane wave with respect to its angle of incidence. The simple way of expressing the higher order excitiations is what enables us to calculate the higher order responses conveniently.

3. Expansion of the scattered waves in terms of plane waves.

A. Expansion of the response of cylinder A in a neighborhood of cylinder B.

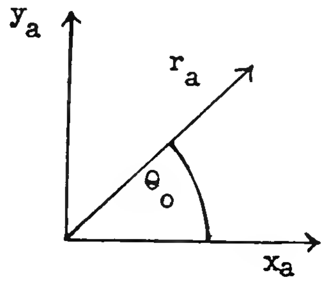

FIGURE 3

Let us consider what happens when a plane wave is incident upon A. The wave function for a plane wave is (Figure 3).

$$U_i = e^{ik(x_a \cos\theta_o + y_a \sin\theta_o)} \tag{2}$$

The wave scattered by A in response to the plane wave is represented by an asymptotic solution of the reduced wave equation.

(3) D.E. $$(\Delta + k^2)\, U = 0 \, .$$

The boundary conditions are such that the diffraction problem for each cylinder, and for the combination, is well posed. They may be, for example,

(4) B.C. $$\begin{cases} \text{a)} & U = 0 \\ \text{or b)} & \dfrac{\partial U}{\partial n} = 0 \\ \text{or c)} & CU + D\dfrac{\partial U}{\partial n} = 0 \end{cases}$$

where $\frac{\partial U}{\partial n}$ is the normal derivative of U and C and D are constants. Alternatively, one or both of the cylinders may be filled with dielectric materials.

The radiation condition for an outgoing wave i.e.;

(5) $$\lim_{r \to \infty} r^{1/2} \left[\frac{\partial U}{\partial r} - ikU \right] = 0$$

is imposed on all scattered fields which occur.

The solution of (3) has the form

(6) $$U = U_i + U_a$$

where U_i is the incident field and U_a is the field scattered by cylinder A. We assume that U_a may be represented in the asymptotic form

(7) $$U_a = \frac{e^{ikr_a}}{\sqrt{r_a}} \left[f_o^{\,a_o}(\theta_a, \theta_o) + \frac{f_1^{\,a_o}}{r_a}(\theta_a, \theta_o) + O(r_a^{-2}) \right] .$$

The letter "a" signifies that the variable in question refers to cylinder A, and the superscript "$^-$" signifies that the pattern is unperturbed.

If, as we have assumed, B is sufficiently small in relation to its distance from A, then the wave scattered by A is practically a plane wave in the neighborhood of B, and we may imagine such a wave incident on B. We can demonstrate an explicit representation of this approximately plane wave by expanding U_a in a neighborhood of B. We shall express the expansion in powers of $d^{-1/2}$ in a rectangular coordinate system with origin at the center of B' and carry out the calculation up to order $d^{-3/2}$. The calculation proceeds as follows:

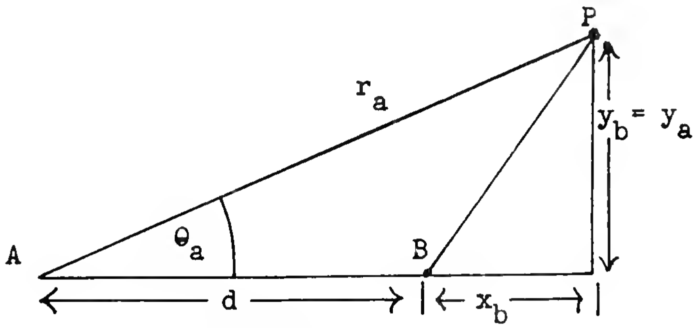

FIGURE 4

Let P be a point in the neighborhood of B.(Figure 4). Let r = the distance from the axis of A' to P. We see from Figure 4 that $r_a = \sqrt{(d+x_b)^2 + y_b^2}$ and we shall henceforth omit the subscripts "a" and "b" as a matter of convenience. Then

$$(8) \quad e^{ikr} = e^{ik(d+x)}\left[1 + \frac{iky^2}{2d} + \frac{ikxy^2 - 1/4\, k^2y^4}{d^2} + \mathcal{O}(d^{-3})\right]$$

$$(9) \quad r^{-1/2} = (d^2 + 2xd + x^2 + y^2)^{-1/4} = \frac{1}{d^{1/2}} - \frac{x}{2d^{3/2}} + \mathcal{O}(d^{-5/2})$$

$$(10) \quad r^{-3/2} = d^{-3/2}\left(1 - \frac{3}{2}\frac{x}{d} + \mathcal{O}(d^{-2})\right) = \frac{1}{d^{3/2}} + \mathcal{O}(d^{-5/2})$$

$$(10a)\qquad \theta - \theta_o = \arctan \frac{y}{d+x} = \frac{y}{d} - \frac{xy}{d^2} + \mathcal{O}(d^{-3})$$

$$(10b)\qquad f_n^{ao}(\theta,\theta_o) = f_n^{ao}(0,\theta_o) + \left[D_\theta f_n^{ao}(0,\theta_o)\right]\theta + \left[D_\theta^2 f_n^{ao}(0,\theta_o)\right]\frac{\theta^2}{2} + \dots\dots\dots,$$

where $D_\theta \equiv \frac{\partial}{\partial\theta}$. Using (10a), we can rewrite (10b) in the form

$$(11)\qquad f_n^{ao}(\theta,\theta_o) = f_n^{ao}(0,\theta_o) + \left[D_\theta f_n^{ao}(0,\theta_o)\right]\frac{y}{d} - \left[D_\theta f_n^{ao}(0,\theta_o)\right]\frac{xy}{d^2} + \left[D_\theta^2 f_n^{ao}(0,\theta_o)\right]\frac{y^2}{2d^2} + \mathcal{O}(d^{-3})$$

Substituting (8) - (11) into (7), we obtain

$$U_a^o = e^{ik(d+x)}\left[1 + \frac{iky^2}{2d} + \frac{ikxy^2 - 1/4\, k^2y^4}{d^2} + \mathcal{O}(d^{-3})\right]$$

$$\left\{ \left[\frac{1}{d^{1/2}} - \frac{x}{2d^{3/2}} + \mathcal{O}(d^{-5/2})\right] \left[f_o^{ao}(0,\theta_o) + D_\theta f_o^{ao}(0,\theta_o)\frac{y}{d} + \mathcal{O}(d^{-2})\right] + \frac{1}{d^{3/2}}\left[f_1^{ao}(0,\theta_o) + \mathcal{O}(d^{-1})\right] \right\}$$

$$(12)\qquad U_a^o = e^{ik(d+x)}\left[\frac{f_o^{ao}(0,\theta_o)}{d^{1/2}} + \frac{1/2(-x + iky^2)\, f_o^{ao}(0,\theta_o) + yD_\theta f_o^{ao}(0,\theta_o) + f_1^{ao}(0,\theta_o)}{d^{3/2}} + \mathcal{O}(d^{-5/2})\right]$$

We have thus obtained the field scattered by A as it appears in a neighborhood of B. We must now carry out the corresponding expansion for

the field initially scattered by B. Note that the leading term of (12) is a plane wave, i.e., a constant multiple of e^{ikx}, as we explained earlier.

B. Expansion in a neighborhood of A for the wave scattered by B.

The same type of procedure as that used above gives us the expansion in a neighborhood of A of the wave initially scattered by B. The expression here corresponding to (12) is:

$$(13) \quad U_b^o = e^{ik(d-x)}\left[\frac{f_o^{bo}(\pi,\theta_o)}{d^{1/2}} + \frac{1/2(x+iky^2)\, f_o^{bo}(\pi,\theta_o) - yD_\theta f_o^{bo}(\pi,\theta_o) + f_1^{bo}(\pi,\theta_o)}{d^{3/2}} + \mathcal{O}(d^{-5/2})\right]$$

The unperturbed fields scattered by either cylinder, (12) or (13), are the excitations of the other cylinder. We assumed, to begin with, that we knew the unperturbed response of each cylinder to a plane wave excitation for all angles of incidence. If (12) and (13) were plane waves, we could calculate the responses for the second scattering.* (12) and (13), however, are not plane waves. But this impediment does not prevent the calculation of the effect of further scattering. The reason is that (12) and (13) may be represented in terms of plane waves by appropriate substitutions. This representation, which will enable us to calculate successive scattering, will now be given.

C. Expression of the scattered waves in terms of plane waves.

We can reduce the further scattering of singly scattered waves to the scattering of plane waves by expressing (12) and (13) in terms of

*This would be the method used in [3].

plane waves and derivatives of plane waves. Noting that a plane wave is represented by

$$(14) \qquad v(\theta_o) = e^{ik(x \cos \theta_o + y \sin \theta_o)}$$

we observe that

$$(15) \qquad ikye^{ikx} = v_{\theta_o}(0)$$

$$(16) \qquad -ikye^{-ikx} = v_{\theta_o}(\pi)$$

$$(17) \qquad ik(-x+iky^2)e^{ikx} = v_{\theta_o\theta_o}(0)$$

$$(18) \qquad ik(x+iky^2)e^{-ikx} = v_{\theta_o\theta_o}(\pi)$$

Substitution of (15) and (17) in (12) and of (16) and (18) in (13) yields the following representation of the scattered fields in terms of plane waves:

$$(19) \quad U_a^o = e^{ikd}\left[\frac{v(o)f_o^{ao}(0,\theta_o)}{d^{1/2}} + \frac{\frac{1}{2ik}\left[D^2_{\theta_o} v(0)\right] f_o^{ao}(0,\theta_o) + \frac{1}{ik}\left[D_{\theta_o} v(0)\right] D_\theta f_o^{ao}(0,\theta_o) + v(0)\, f_1^{ao}(0,\theta_o)}{d^{3/2}} + \mathcal{O}(d^{-5/2})\right]$$

$$(20) \quad U_b^o = e^{ikd}\left[\frac{v(\pi)f_o^{bo}(\pi,\theta_o)}{d^{1/2}} + \frac{\frac{1}{2ik}\left[D^2_{\theta_o} v(\pi)\right] f_o^{bo}(\pi,\theta_o) + \frac{1}{ik}\left[D_{\theta_o} v(\pi)\right] D_\theta f_o^{bo}(\pi,\theta_o) + v(\pi)\, f_1^{bo}(\pi,\theta_o)}{d^{3/2}} + \mathcal{O}(d^{-5/2})\right]$$

D. Elimination of f_1

We note that the numerators of (19) and (20) are sums of terms, consisting of plane waves and their derivatives, namely the v's, and coefficients which are independent of θ, namely the f's. We see that these formulas are expressed in terms of both f_o and f_1. (For the meaning of f_o and f_1 see (7) above). An advantage would result from the elimination of f_1, since we could then express the result in terms of the scattering amplitude of the far field without having to know the scattering amplitudes of further asymptotic terms.

We eliminate f_1 by expressing it in terms of f_o. This can be done by means of a recursion formula. The recursion is obtained by substitution into (3) of the assumed asymptotic* representation (7), of any radiating solution of the reduced wave equation. When we equate the corresponding inverse powers of r, we find that

$$f_n = \frac{1}{2ikn}\left[(n - \tfrac{1}{2})^2 f_{n-1} + D_\theta^2 f_{n-1}\right] . \tag{21}$$

This recursion is useful, also, for calculations of higher degree than we are considering here. Since we want to express f_1 in terms of f_o, we need use it only for the value n = 1.

$$f_1 = \frac{1}{2ik}\left[\frac{1}{4} f_o + D_\theta^2 f_o\right] \tag{22}$$

This is a different type of recursion from that obtained for large k by Keller, Lewis, and Seckler[8] although it is similar in form.

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*The representation is asymptotic for large r, θ and k being held fixed.

If we now substitute (22) into (19) and (20), we obtain:

$$(23) \qquad U_a^o = e^{ikd}\left[\frac{v(0) f_o^{ao}(0,\theta_o)}{d^{1/2}} + \frac{\frac{1}{2ik}\left[D_{\theta_o}^2 v(0)\right] f_o^{ao}(0,\theta_o) + \frac{1}{ik}\left[D_{\theta_o} v(0)\right] D_\theta f_o^{ao}(0,\theta_o) + \frac{1}{2ik} v(0)\left[\frac{1}{4} f_o^{ao}(0,\theta_o) + D_\theta^2 f_o^{ao}(0,\theta_o)\right]}{d^{3/2}} + \mathcal{O}(d^{-5/2})\right]$$

$$(24) \qquad U_b^o = e^{ikd}\left[\frac{v(0) f_o^{bo}(\pi,\theta_o)}{d^{1/2}} + \frac{\frac{1}{2ik}\left[D_{\theta_o}^2 v(\pi)\right] f_o^{bo}(\pi,\theta_o) + \frac{1}{ik}\left[D_{\theta_o} v(\pi)\right] D_\theta f_o^{bo}(\pi,\theta_o) + \frac{1}{2ik} v(\pi)\left[\frac{1}{4} f_o^{bo}(\pi,\theta_o) + D_\theta^2 f_o^{bo}(\pi,\theta_o)\right]}{} + \mathcal{O}(d^{-5/2})\right].$$

The above expressions (23) and (24) represent the responses of the cylinders to the original incident plane wave and these responses are given near the other cylinder in terms of plane waves. (23) and (24) are also excitations for the second scattering. Successive application of these formulas will yield the desired degree of interaction.

4. Calculation of the Interaction.

We have expressed in (23) and (24) the fields singly scattered by A and by B, as they appear in a neighborhood of the second scatterer and have moreover expressed them in terms of plane waves and derivatives of

plane waves. We may now imagine this combination of plane waves and their derivatives to be incident upon the second scatterer. The linearity of these expressions enables us to say that the responses of A and B to incident derivatives of plane waves are equal to the derivatives of the responses of A and B to the incident plane waves. Since we already know, by assumption, the unperturbed responses of A and B to an incident plane wave, we have reduced the second scattering to the previous case, namely the first scattering.

If we carry out this process to the extent of three successive scatterings, we can obtain interaction terms of degrees $d^{-1/2}$, d^{-1} and $d^{-3/2}$. The terms of degree $d^{-1/2}$ result from double scattering, those of degree d^{-1} result from triple scattering, and those of degree $d^{-3/2}$ result partly from double and partly from quadruple scattering.

For the sake of simplicity, the following results will be expressed in a normalized coordinate system (Figure 5) with a common origin midway between the origins of the coordinate systems located in the scatterers. As a matter of convenience, we shall omit the subscript zero from the f's.

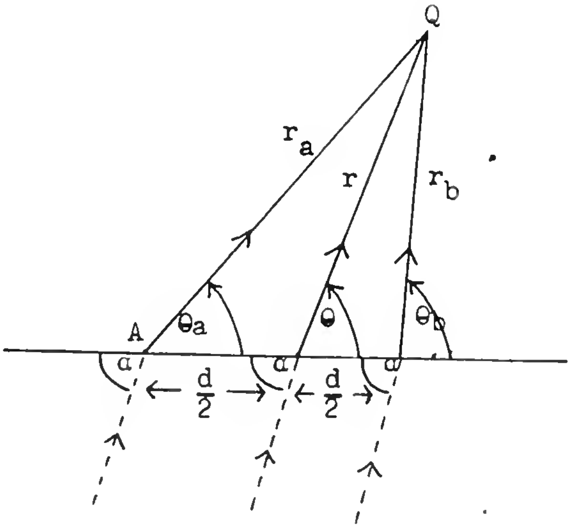

FIGURE 5

Let α be the angle of incidence of the original plane wave. The perturbed patterns will then be

$$(25)\quad f^{a}(\theta,\alpha) = e^{-ik\frac{d}{2}\cos\alpha} f^{ao}(\theta,\alpha)$$

$$+ \frac{e^{ik\frac{d}{2}\cos\alpha}\, e^{ikd} f^{bo}(\pi,\alpha) f^{ao}(\theta,\pi)}{d^{1/2}}$$

$$+ \frac{e^{-ik\frac{d}{2}\cos\alpha}\, e^{ik2d} f^{ao}(0,\alpha) f^{bo}(\pi,0) f^{ao}(\theta,\pi)}{d}$$

$$+ \frac{e^{ik\frac{d}{2}\cos\alpha}\, e^{ik3d}}{d^{3/2}} f^{bo}(\pi,\alpha) f^{ao}(o,\pi) f^{bo}(\pi,o) f^{ao}(\theta,\pi)$$

$$+ \frac{e^{ik\frac{d}{2}\cos\alpha}}{d^{3/2}} \frac{e^{ikd}}{2ik} \Bigg(f^{bo}(\pi,\alpha) D^{2}_{\theta_o} f^{ao}(\theta,\pi) + 2D_{\theta} f^{bo}(\pi,\alpha) D_{\theta_o} f^{ao}(\theta,\pi)$$

$$+ \left[\frac{1}{4} f^{bo}(\pi,\alpha) + D^{2}_{\theta} f^{bo}(\pi,\alpha)\right] f^{ao}(\theta,\pi) \Bigg)$$

$$+ \mathcal{O}(d^{-5/2})$$

and

$$(26)\quad f^{b}(\theta,\alpha) = e^{ik\frac{d}{2}\cos\alpha} f^{bo}(\theta,\alpha) + \frac{e^{-ik\frac{d}{2}\cos\alpha}\, e^{ikd}}{d^{1/2}} f^{ao}(\theta,\alpha) f^{bo}(\theta,o)$$

$$+ \frac{e^{ik\frac{d}{2}\cos\alpha}\, e^{ik2d}}{d} f^{bo}(\pi,\alpha) f^{ao}(o,\pi) f^{bo}(\theta,o)$$

$$+ \frac{e^{-ik\frac{d}{2}\cos\alpha}\, e^{ik3d}}{d^{3/2}} f^{ao}(o,\alpha) f^{bo}(\pi,o) f^{ao}(o,\pi) f^{bo}(\theta,0)$$

$$+ \frac{e^{-ik\frac{d}{2}\cos\alpha}\, e^{ikd}}{d^{3/2}} \Bigg(f^{ao}(o,\alpha) D^{2}_{\theta_o} f^{bo}(\theta,o) + 2D_{\theta} f^{ao}(o,\alpha) D_{\theta_o} f^{bo}(\theta,o)$$

$$+ \left[\frac{1}{4} f^{ao}(o,\alpha) + D^{2}_{\theta} f^{ao}(o,\alpha)\right] f^{bo}(\theta,o) \Bigg)$$

$$+ \mathcal{O}(d^{-5/2})$$

We note that the successive powers of $d^{-1/2}$ in (25) and (26) represent the various degrees of interaction. The scattered far fields resulting from the interaction will have the form

$$U_a = \frac{e^{ik(r + \frac{d}{2}\cos\theta)}}{\sqrt{r}} f^a(\theta,\alpha) \tag{27}$$

and

$$U_b = \frac{e^{ik(r - \frac{d}{2}\cos\theta)}}{\sqrt{r}} f^b(\theta,\alpha) \tag{28}$$

where $f^a(\theta,\alpha)$ and $f^b(\theta,\alpha)$ are the perturbed scattering amplitudes given by (25) and (26). The sum of the scattered fields has the following far field representation which is the sum of (27) and (28).

$$U_s = U_a + U_b = \frac{e^{ikr}}{\sqrt{r}} \sqrt{\frac{2}{\pi k}}\; e^{-i\frac{\pi}{4}} F(\theta,\alpha) \tag{29}$$

where $F(\theta,\alpha)$ is the scattering amplitude of the combination of cylinders. We see by comparison with (27) and (28) that

$$F(\theta,\alpha) = \sqrt{\frac{\pi k}{2}}\; e^{i\frac{\pi}{4}} \left(e^{ik\frac{d}{2}\cos\theta} f^a(\theta,\alpha) + e^{-ik\frac{d}{2}\cos\theta} f^b(\theta,\alpha) \right) \tag{30}$$

A more explicit formula for the far field is obtained by combining (25), (26), (29) and (30). The far field can then be written as follows:

$$(31)\quad u \cong \frac{e^{ikr}}{\sqrt{r}}\Bigg\{ e^{ik\frac{d}{2}(\cos\theta-\cos\alpha)} f^{ao}(\theta,\alpha) + e^{-ik\frac{d}{2}(\cos\theta-\cos\alpha)} f^{bo}(\theta,\alpha)$$

$$+ \frac{e^{ikd}}{d^{1/2}}\left[e^{ik\frac{d}{2}(\cos\theta+\cos\alpha)} f^{bo}(\pi,\alpha) f^{ao}(\theta,\pi) + e^{-ik\frac{d}{2}(\cos\theta+\cos\alpha)} f^{ao}(o,\alpha) f^{bo}(\theta,o)\right]$$

$$+ \frac{e^{ik2d}}{d}\Big[e^{ik\frac{d}{2}(\cos\theta-\cos\alpha)} f^{ao}(o,\alpha) f^{bo}(\pi,o) f^{ao}(\theta,\pi)$$

$$+ e^{-ik\frac{d}{2}(\cos\theta-\cos\alpha)} f^{bo}(\pi,\alpha) f^{ao}(o,\pi) f^{bo}(\theta,o)\Big]$$

$$+ \frac{e^{ik3d}}{d^{3/2}}\Big[e^{ik\frac{d}{2}(\cos\theta+\cos\alpha)} f^{bo}(\pi,\alpha) f^{ao}(o,\pi) f^{bo}(\pi,o) f^{ao}(\theta,\pi)$$

$$+ e^{-ik\frac{d}{2}(\cos\theta+\cos\alpha)} f^{ao}(o,\alpha) f^{bo}(\pi,o) f^{ao}(o,\pi) f^{bo}(\theta,o)\Big]$$

$$+ \frac{e^{ikd}}{2ikd^{3/2}}\Bigg[e^{ik\frac{d}{2}(\cos\theta+\cos\alpha)}\Bigg(f^{bo}(\pi,\alpha) D^2_{\theta_o} f^{ao}(\theta,\pi) + 2D_\theta f^{bo}(\pi,\alpha) D_{\theta_o} f^{ao}(\theta,\pi)$$

$$+ \left\{\frac{1}{4} f^{bo}(\pi,\alpha) + D^2_\theta f^{bo}(\pi,\alpha)\right\} f^{ao}(\theta,\pi)\Bigg)$$

$$+ e^{-ik\frac{d}{2}(\cos\theta+\cos\alpha)}\Bigg(f^{ao}(o,\alpha) D^2_{\theta_o} f^{bo}(\theta,o) + 2D_\theta f^{ao}(o,\alpha) D_{\theta_o} f^{bo}(\theta,o)$$

$$+ \left\{\frac{1}{4} f^{ao}(o,\alpha) + D^2_\theta f^{ao}(o,\alpha)\right\} f^{bo}(\theta,o)\Bigg)\Bigg]$$

$$+ \mathcal{O}(d^{-5/2})\Bigg\}$$

We have presented in (31) a relation between the scattered far field of the combination and those of the component cylinders. We wish to point out that a similar relation holds between the corresponding fields at all points of space. But this relation will not be detailed here.

5. A special case. Scattering by two conducting circular cylinders.

The abstract relations obtained above can be verified in the special case of scattering by two parallel conducting circular cylinders A and B with radii a and b respectively. If the electric field vector is parallel to the axis of the cylinders, we are confronted with the familiar boundary value problem

D.E. $(\Delta + k^2)u = o$

(32) B.C. $u = o$ on the boundaries of the cylinders.

(33) R.C. $\lim\limits_{r\to\infty} \sqrt{r}\left(\frac{\partial U}{\partial r} - ikU\right) = o$

Solution of this problem for each cylinder in isolation requires the determination of coefficients a_n^o and b_n^o in (34) and (35), below

$$U_a^o = \sum_{n=-\infty}^{\infty} a_n^o H_n^{(1)}(kr_a)e^{in(\theta_a-\alpha)} \tag{34}$$

and

$$U_b^o = \sum_{n=-\infty}^{\infty} b_n^o H_n^{(1)}(kr_b)e^{in(\theta_b-\alpha)} \tag{35}$$

Expansion of the incident plane wave in terms of cylindrical waves by means of the Fourier-Bessel expansion [9] and application of the boundary conditions(32) yields

$$(36)\quad U_a^o = \sum_{n=-\infty}^{\infty} -i^n \frac{J_n(ka)}{H_n^{(1)}(ka)} H_n^{(1)}(kr_a)\, e^{in(\theta_a-\alpha)}$$

$$(37)\quad U_b^o = \sum_{n=-\infty}^{\infty} -i^n \frac{J_n(kb)}{H_n^{(1)}(kb)} H_n^{(1)}(kr_b)\, e^{in(\theta_b-\alpha)}$$

for the unperturbed scattered fields, as is well known. Using the asymptotic form of $H^{(1)}$ for large argument

$$H_n^{(1)}(kr) \sim \frac{e^{ikr}}{\sqrt{r}}\, e^{-i\frac{\pi}{4}} \sqrt{\frac{2}{\pi k}}\; i^{-n}$$

we see that the far fields are

$$(38)\quad U_a^o = \frac{e^{ikr_a}}{\sqrt{r_a}}\, e^{-i\frac{\pi}{4}} \sqrt{\frac{2}{\pi k}}\; (-1) \sum_{n=-\infty}^{\infty} \frac{J_n(ka)}{H_n^{(1)}(ka)}\, e^{in(\theta_a-\alpha)}$$

$$(39)\quad U_b^o = \frac{e^{ikr_b}}{\sqrt{r_b}}\, e^{-i\frac{\pi}{4}} \sqrt{\frac{2}{\pi k}}\; (-1) \sum_{n=-\infty}^{\infty} \frac{J_n(kb)}{H_n^{(1)}(kb)}\, e^{in(\theta_b-\alpha)}$$

Normalizing our coordinate system to a new origin midway between the origins in the scatterers as we did above in the abstract treatment, we obtain

$$(40)\quad U_a^o = \frac{e^{ik(r+\frac{d}{2}\cos\theta)}}{\sqrt{r}}\, e^{-ik\frac{d}{2}\cos\alpha}\, e^{-i\frac{\pi}{4}} \sqrt{\frac{2}{\pi k}}\; (-1) \sum_{n=-\infty}^{\infty} \frac{J_n(ka)}{H_n^{(1)}(ka)}\, e^{in(\theta-\alpha)},$$

and

$$(41)\quad U_b^o = \frac{e^{ik(r-\frac{d}{2}\cos\theta)}}{\sqrt{r}}\, e^{+ik\frac{d}{2}\cos\alpha}\, e^{-i\frac{\pi}{4}} \sqrt{\frac{2}{\pi k}}\; (-1) \sum_{n=-\infty}^{\infty} \frac{J_n(kb)}{H_n^{(1)}(kb)}\, e^{in(\theta-\alpha)},$$

from which we observe that the unperturbed patterns are respectively

$$(42)\quad f_o^{ao}(\theta,\alpha) = e^{-i\frac{\pi}{4}}\sqrt{\frac{2}{\pi k}}\ (-1)\sum_{n=-\infty}^{\infty}\frac{J_n(ka)}{H_n^{(1)}(ka)}\ e^{in(\theta-\alpha)}$$

$$(43)\quad f_o^{bo}(\theta,\alpha) = e^{-i\frac{\pi}{4}}\sqrt{\frac{2}{\pi k}}\ (-1)\sum_{n=-\infty}^{\infty}\frac{J_n(kb)}{H_n^{(1)}(kb)}\ e^{in(\theta-\alpha)}$$

The perturbed patterns can be obtained by the following procedure:

We take the multiple scattering viewpoint, as we did in the abstract case. Each excitation elicits a response. The responses to the initial plane wave have the form (34) and (35). These responses, in turn, excite the second cylinder, eliciting new responses. Unlike the procedure of the abstract treatment, however, these excitations are not expressed in terms of plane waves, nor are the responses expressed in an asymptotic form.

At all stages intermediate between the initial plane wave excitation and the evaluation of the far field response of the combination, the excitations will have the general form

$$(44)\quad \sum_{p=-\infty}^{\infty} c_p\, H_p^{(1)}\,(kr)\ e^{ip\theta}$$

and the responses will have the general form

$$(45)\quad \sum_{q=-\infty}^{\infty} d_q H_q^{(1)}(kr)\ e^{iq\theta}$$

The proper adjustments must be made in (44) and (45) for angles of incidence and the fields must be expressed in their proper coordinate

systems. The coefficients C_p and d_q are determined successively for each scattering by the boundary conditions on the cylinders which are, in this case, that the sum of the excitation and its response is, for each cylinder, zero on the boundary of that cylinder. The application of the boundary conditions requires that each excitation be represented in the same coordinate system as its response. This is accomplished by the application of addition theorems for cylindrical waves [9]. The process* is continued until the desired degree of scattering is obtained. Application of this method yields, up to terms of degree $d^{-3/2}$, the following perturbed patterns:

(Equations on pages 22 and 23)

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*Throughout this calculation we follow the procedure of Twersky [2]; but, unlike that author, we do not neglect any terms of order $d^{-3/2}$, in computing the final form of the result for large d.

$$
\begin{aligned}
(46)\quad f_o^a(\theta,\alpha) = {} & e^{-ik\frac{d}{2}\cos\alpha}\, e^{-i\frac{\pi}{4}} \sqrt{\frac{2}{\pi k}} \sum_n (-1) \frac{J_n(ka)}{H_n^{(1)}(ka)}\, e^{in(\theta-\alpha)} \\
& + \frac{e^{ik\frac{d}{2}\cos\alpha}\, e^{ikd}}{d^{1/2}}\, e^{-i\frac{\pi}{2}} \left(\frac{2}{\pi k}\right) \sum_n (-1)^n e^{-in\alpha} \frac{J_n(kb)}{H_n^{(1)}(kb)} \sum_{n'} \frac{J_{n'}(ka)}{H_{n'}^{(1)}(ka)}\, e^{in'(\theta-\pi)} \\
& + \frac{e^{-ik\frac{d}{2}\cos\alpha}\, e^{ik2d}}{d}\, e^{-i\frac{3\pi}{4}} \left(\frac{2}{\pi k}\right)^{3/2} \sum_n (-1)\, e^{-in\alpha} \frac{J_n(ka)}{H_n^{(1)}(ka)} \sum_{n'} (-1)^{n'} \frac{J_{n'}(kb)}{H_{n'}^{(1)}(kb)} \sum_{n''} \frac{J_{n''}(ka)}{H_{n''}^{(1)}(ka)}\, e^{in''(\theta-\pi)} \\
& + \frac{e^{ik\frac{d}{2}\cos\alpha}\, e^{ik3d}}{d^{3/2}}\, e^{-i\pi} \left(\frac{2}{\pi k}\right)^{2} \sum_n (-1)^n e^{-in\alpha} \frac{J_n(kb)}{H_n^{(1)}(kb)} \sum (-1)^{n'} \frac{J_{n'}(ka)}{H_{n'}^{(1)}(ka)} \\
& \qquad \sum_{n''} (-1)^{n''} \frac{J_{n''}(kb)}{H_{n''}^{(1)}(kb)} \sum_{n'''} \frac{J_{n'''}(ka)}{H_{n'''}^{(1)}(ka)}\, e^{in'''(\theta-\pi)} \\
& + \frac{e^{ik\frac{d}{2}\cos\alpha}\, e^{ikd}}{d^{3/2}}\, \frac{e^{-i\frac{\pi}{2}}}{2ik} \left(\frac{2}{\pi k}\right) \sum_n (-1)\,(-1)^n e^{-in\alpha} \frac{J_n(kb)}{H_n^{(1)}(kb)} \sum_{n'} \left[(n-n')^2 - \frac{1}{4}\right] \frac{J_{n'}(ka)}{H_{n'}^{(1)}(ka)}\, e^{in'(\theta-\pi)} \\
& + \mathcal{O}(d^{-5/2})
\end{aligned}
$$

$$(47)\quad f_o^b(\theta,\alpha) = e^{ik\frac{d}{2}\cos\alpha}\, e^{-i\frac{\pi}{4}} \sqrt{\frac{2}{\pi k}} \sum (-1)\, \frac{J_n(kb)}{H_n^{(1)}(kb)}\, e^{in(\theta-\alpha)}$$

$$+ \frac{e^{-ik\frac{d}{2}\cos\alpha}\, e^{ikd}}{d^{1/2}}\, e^{-i\frac{\pi}{2}} \left(\frac{2}{\pi k}\right) \sum_n e^{-in\alpha} \frac{J_n(ka)}{H_n^{(1)}(ka)} \sum_{n'} \frac{J_{n'}(kb)}{H_{n'}(kb)}\, e^{in'\theta}$$

$$+ \frac{e^{ik\frac{d}{2}\cos\alpha}\, e^{ik2d}}{d}\, e^{-i\frac{3\pi}{4}} \left(\frac{2}{\pi k}\right)^{3/2} \sum_n (-1)(-1)^n\, e^{-in\alpha} \frac{J_n(kb)}{H_n^{(1)}(kb)} \sum_{n'} (-1)^{n'} \frac{J_{n'}(ka)}{H_{n'}^{(1)}(ka)} \sum_{n''} \frac{J_{n''}(kb)}{H_{n''}^{(1)}(kb)}\, e^{in''\theta}$$

$$+ \frac{e^{-ik\frac{d}{2}\cos\alpha}\, e^{ik3d}}{d^{3/2}}\, e^{-i\pi} \left(\frac{2}{\pi k}\right)^2 \sum_n e^{-in\alpha} \frac{J_n(ka)}{H_n^{(1)}(ka)} \sum_{n'} (-1)^{n'} \frac{J_{n'}(kb)}{H_{n'}^{(1)}(kb)}$$

$$\sum_{n''} (-1)^{n''} \frac{J_{n''}(ka)}{H_{n''}^{(1)}(ka)} \sum_{n'''} \frac{J_{n'''}(kb)}{H_{n'''}^{(1)}(kb)}\, e^{in'''\theta}$$

$$+ \frac{e^{-ik\frac{d}{2}\cos\alpha}\, e^{ikd}}{d^{3/2}}\, \frac{e^{-i\frac{\pi}{2}}}{2ik} \left(\frac{2}{\pi k}\right) \sum_n (-1)\, e^{-in\alpha} \frac{J_n(ka)}{H_n^{(1)}(ka)} \sum_{n'} \left[(n'-n)^2 - \frac{1}{4}\right] \frac{J_{n'}(kb)}{H_{n'}^{(1)}(kb)}\, e^{in'\theta}$$

$$+ \mathcal{O}(d^{-5/2})$$

Calculations of a similar nature have been carried out by Twersky[2] for two identical cylinders with more general boundary conditions. Comparison of (46) and (47) with formula (6) of Twersky's paper show that the non-interaction terms in our result agree with Twersky's first term and our terms of degree $d^{-1/2}$ agree with Twersky's second term. Our term of degree d^{-1} agrees with the third term in Twersky's paper and also with an equivalent corresponding term in a report[1] by Twersky on which his paper was based. Our terms of degree $d^{-3/2}$ do not agree with Twersky's fourth term, but we should not expect such agreement.

Twersky's terms are the asymptotic representations of successive orders of scattering, that is successive bounces. Our successive powers of $d^{-1/2}$, however, do not correspond to successive bounces. Our term of degree $d^{-3/2}$, for example, contains contributions from the second bounce as well as the fourth bounce.

The procedure used above for obtaining (46) and (47) is long and tedious. These results need not be obtained by that procedure. The use of the abstract formulas (25) and (26) simplify the calculation considerably and should do the same in any other case where the scattering problems for the component cylinders are separable. All we need do is substitute (42) and (43) into (25) and (26) respectively. The results follow immediately and agree with (46) and (47) respectively.

6. <u>An Explanation of the Result</u>

The expression (31) for the scattered far field of the combination of two cylinders appears, at first glance, to be complicated. However, it is not as abstruse as it seems. A closer examination of these expressions

reveals the significance of these various terms and factors. We note first that the terms are grouped in increasing orders of accuracy. We note also that they are grouped in pairs. The first members of each pair represent fields ultimately scattered by cylinder A while the second members of the pairs represent fields ultimately scattered by cylinder B. The first pair corresponds to single scattering while the other pairs correspond to multiple scattering. The factors $e^{\pm ik\frac{d}{2}\cos\theta}$ represent the phase differences for the scatterers relative to the point of observation while the factors $e^{\pm ik\frac{d}{2}\cos\alpha}$ take into account the phase of the incident wave at the center of a scatterers when the incident wave has zero phase at the origin. The f's containing a θ dependence are scattering patterns whereas the f's containing specific values for θ are excitation factors accumulated in the multiple scattering. We note also that the factors of the form e^{iknd} where n = 0, 1, 2, 3 refer to the increase ikd in the phase of a wave in going from one scatterer to another and that n signifies the number of bounces.

As an illustration of the above remarks, let us consider some of the terms in more detail. The term $\frac{eikr}{\sqrt{r}}\, e^{ik\frac{d}{2}(\cos\theta - \cos\alpha)} f^{ao}(\theta,\alpha)$ would occur in the case of no interaction, i.e. in the limit of infinite spacing. The factor $\exp\frac{ikd}{2}(\cos\theta)$ takes account of the fact that the origin is not at the center of A. The positive sign preceding cos θ shows that the scattered wave came from cylinder A. The negative sign in the exponent of the factor $e^{-ik\frac{d}{2}\cos\alpha}$ shows that cylinder A received the milial excitation. It gives the phase of the incident wave at A. The term

$$\frac{e^{ikr}}{\sqrt{r}}\, e^{ik\frac{d}{2}(\cos\theta + \cos\alpha)} f^{bo}_{(\pi,\alpha)} f^{ao}_{(\theta,\pi)}$$

differs from the first term in the following respects. The positive sign in the factor $\exp(+\frac{ikd}{2}\cos\alpha)$ shows that cylinder B received the original excitation as does the factor $f^{bo}_{(\pi,\alpha)}$. The phase factor e^{ikd} represents the increase of the phase of the wave in going from B to A.

We shall now explain a typical term of the last square bracket. The first term in the last square bracket of (31) represents the scattering by A of a term of order $d^{-3/2}$ initially scattered by B. The phase factor e^{ikd} takes account of the travel of this wave from B to A.

The positive sign preceding $\cos\alpha$ shows that cylinder B was excited initially. The differentiation of the scattering amplitudes shows the effect of a higher order excitation of A by the field initially scattered by B, since the higher order response of B, (which acts as an excitation for A), is representable, near A, as a derivative of a plane wave with respect to angle of incidence.

The explanation of further terms in the final result proceeds on the same lines as the explanations given above. We omit these explanations for the sake of brevity.

7. The total scattering cross-section of a combination of two identical circular cylinders.

We consider a plane wave normally incident on a pair of identical circular cylinders. The circumstances are illustrated in Figure 6.

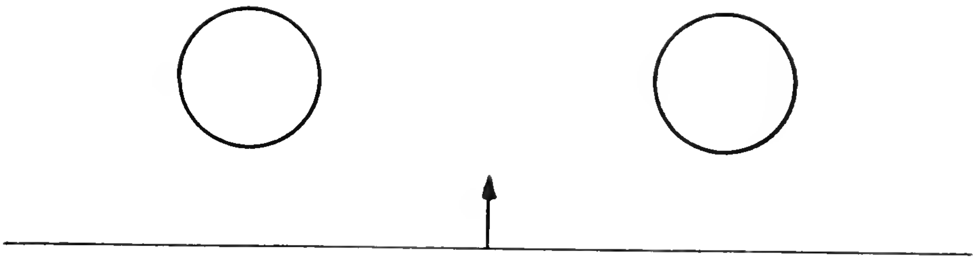

FIGURE 6

The computation of the total scattering cross section σ of the two identical circular cylinders is facilitated by the use of (30) in conjunction with the following well known theorem [12], [13],

$$\sigma = -\frac{4}{k} \operatorname{Re} F(\tfrac{\pi}{2},\tfrac{\pi}{2}) \quad . \tag{48}$$

where $F(\theta,\alpha)$ is the scattering amplitude (30) of the combination. The phase factors are simplified for the values $\theta = \frac{\pi}{2}$, $\alpha = \frac{\pi}{2}$. The fact that the cylinders are identical enables us to write $f^{o} = f^{ao} = f^{bo}$.

Further simplifications result from the geometrical symmetry of the problem, namely

$$f^{o}(o,\pi) = f^{o}(\pi,o),\ f^{o}(\tfrac{\pi}{2},o) = f^{o}(o,\tfrac{\pi}{2}) = f^{o}(\pi,\tfrac{\pi}{2}) = f^{o}(\tfrac{\pi}{2},\pi)$$

The total scattering cross section in terms of the spacing is then

$$(49)\quad \sigma = -\frac{4}{k}\,\mathrm{Re}\left[\sqrt{2\pi k}\; e^{i\frac{\pi}{4}} \left\{ \begin{aligned} & f^{o}\!\left(\tfrac{\pi}{2},\tfrac{\pi}{2}\right) + \frac{e^{ikd}}{d^{1/2}}\left[f^{o}\!\left(o,\tfrac{\pi}{2}\right)\right]^{2} + \frac{e^{ik2d}}{d}\left[f^{o}\!\left(o,\tfrac{\pi}{2}\right)\right]^{2} f^{o}(\pi,o) \\ & + \frac{e^{ik3d}}{d^{3/2}}\left[f^{o}\!\left(o,\tfrac{\pi}{2}\right)\right]^{2}\left[f^{o}(\pi,o)\right]^{2} \\ & + \frac{e^{ikd}}{d^{3/2}}\,\frac{1}{2ik}\left(f^{o}\!\left(o,\tfrac{\pi}{2}\right) D^{2}_{\theta_{o}} f^{o}\!\left(o,\tfrac{\pi}{2}\right) + 2D_{\theta} f^{o}\!\left(o,\tfrac{\pi}{2}\right) D_{\theta_{o}} f^{o}_{\phi}\!\left(o,\tfrac{\pi}{2}\right) \right. \\ & \qquad \left. + \left[\tfrac{1}{4} f^{o}\!\left(o,\tfrac{\pi}{2}\right) + D^{2}_{\theta} f^{o}\!\left(o,\tfrac{\pi}{2}\right)\right] f^{o}\!\left(o,\tfrac{\pi}{2}\right) \right) \\ & + \mathcal{O}\!\left(d^{-5/2}\right) \end{aligned} \right\} \right]$$

8. Additional Remarks

It is clear that interactions of degree greater than $d^{-3/2}$ can be computed by the inclusion of more terms in the expansions used to obtain these results.

It is also clear that the method applies to cases of more than two scatterers, but the computations would be more tedious than in the case of two scatterers. The computations might be simplified by using a " consistency" method employed in [2] rather than tracing the successive scattering in detail. This method involves a steady state point of view. The response of each cylinder is expanded in a neighborhood of each of the other cylinders. Each cylinder will then be excited by the incident plane wave and by an approximately plane wave from each of the other cylinders.

These considerations introduce certain undetermined coefficients which can be determined by imposing the requirement that the field scattered by the various cylinders be consistent with one another. Evaluation of the coefficients will provide the solution.

II. Three Dimensional Case.

1. Statement of the Problem. The method employed previously in the case of multiple scattering of plane waves by two widely spaced cylinders of arbitrary shape can be applied, also to the corresponding three dimensional scalar problem for two bodies of arbitrary shape. The assumptions here, as in the previous case, are that:

1) The spacing of the bodies is large compared to their dimensions.

2) The spacing is large compared to the wavelength of the incident plane wave.

3) The response of each scatterer to a plane wave excitation is known when the scatterer is isolated in space. In other words, the individual unperturbed complex scattering amplitudes of the far fields are known.

The situation is the following:

A plane wave of unit amplitude

$$(1) \quad u = e^{ik(x \sin \theta_o \cos \phi_o + y \sin \theta_o \sin \phi_o + z \cos \theta_o)}$$

where θ_o and ϕ_o are the angles of incidence (see Figure 1), is incident upon the combination of two bodies (Figure 2, page 32).

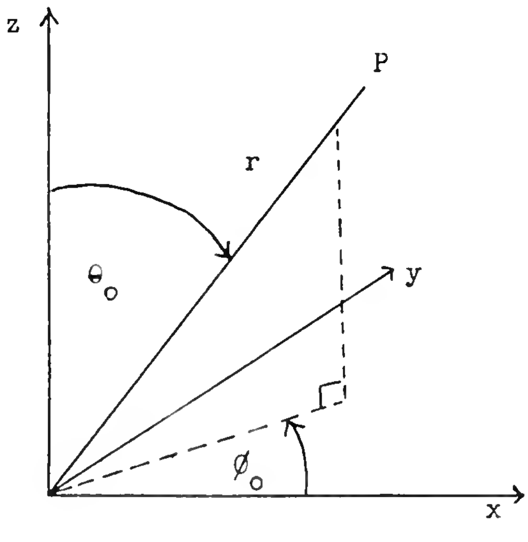

Figure 1

The response of each body in isolation to the incident plane wave is of the form

$$(2) \qquad U_s = \frac{e^{ikr}}{r} \sum_{n=o}^{\infty} \frac{f_n(\theta,\theta_o,\phi,\phi_o)}{r^n}$$

Here we are referring to spherical coordinates, r, θ, ϕ, such that

$$(2a) \qquad \begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

The letters x, y, z are cartesian coordinatesof a point P with respect to a coordinate system located appropriately. The quantities θ_o and ϕ_o which specify the direction of travel of the incident wave can be thought of as the colatitude and longitude respectively of a point on the unit sphere. That point is determined as the intersection with the unit sphere of a line passing through the origin in the direction of travel of the incident wave.

The object is to obtain the perturbed complex scattering amplitude of the combination as a function of the individual unperturbed scattering amplitudes. This can be done, as in the previous case, by first expanding the field scattered by each body in a neighborhood of the other body and then expressing these scattered fields in terms of plane waves and their derivatives. The a priori assumption that the unperturbed responses to an incident plane wave are known permits the calculation of the perturbed responses.

We must establish coordinate systems for the problem. We first circumscribe spheres A' and B' about the respective bodies A and B.

This establishes the origins of the coordinate systems of A and B (Figure 2)

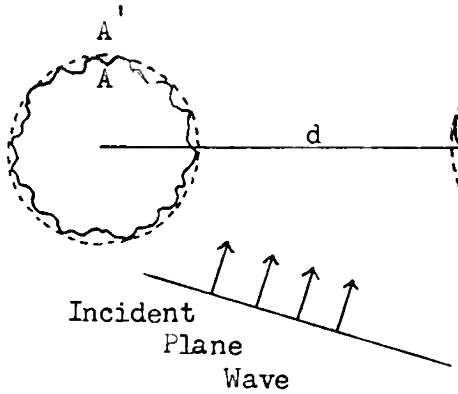

FIGURE 2

The spacing d is defined as the line joining the two centers. We then choose the x_a axis (Figure 3)...i.e. the x axis for the coordinate system of A, to be collinear with d and such that x_a increases in the direction of B.

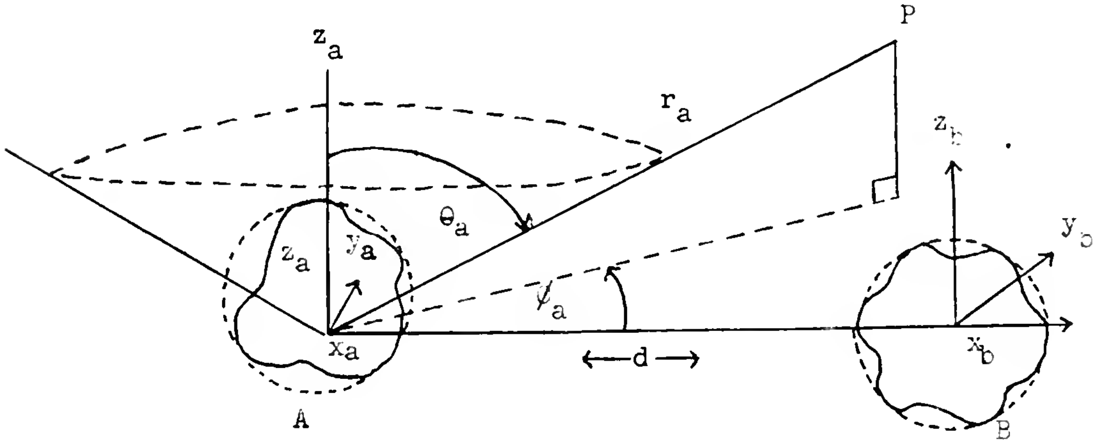

FIGURE 3

The y_a and z_a axes can then be chosen to form a right-handed system and the coordinate system of B is parallel to that of A. (Choice of the z axis along d, instead of the x axis, would be more symmetrical, but has been avoided for reasons which will be explained later.)

2. Expansion of the scattered waves in terms of plane waves.

We proceed to expand the response of each body in a neighborhood of the other body. This expansion yields the following field scattered by A in a neighborhood of B where the cartesian coordinates are in the B system.

$$(3)\quad U^{o}_{s_a} = e^{ik(d+x_b)}\Bigg[\frac{1}{d}f^{ao}_{o}(\tfrac{\pi}{2},\theta_o,0,\phi_o)+\frac{1}{d^2}\Bigg\{\frac{ik(y^2+z^2)}{2}f^{ao}_{o}(\tfrac{\pi}{2},\theta_o,0,\phi_o)-x_b f^{ao}_{o}(\tfrac{\pi}{2},\theta_o,0,\phi_o)$$

$$-zD_\theta f^{ao}_{o}(\tfrac{\pi}{2},\theta_o,0,\phi_o)+yD_\phi f^{ao}_{o}(\tfrac{\pi}{2},\theta_o,0,\phi_o)$$

$$+f^{ao}_{1}(\tfrac{\pi}{2},\theta_o,0,\phi_o)\Bigg\}+\mathcal{O}(d^{-3})\Bigg]$$

In calculating equation (3) the phase of the incident wave at the 'center' of A has been ignored. The appropriate phase factor will be restored in the final result. A similar expansion of the field scattered by B in a neighborhood of A yields

$$(4)\quad U^{o}_{s_b} = e^{ik(d-x_a)}\Bigg[\frac{1}{d}f^{bo}_{o}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o)+\frac{1}{d^2}\Bigg\{\frac{ik(y^2+z^2)}{2}f^{bo}_{o}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o)+x_a f^{bo}_{o}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o)$$

$$-zD_\theta f^{bo}_{o}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o)-yD_\phi f^{bo}_{o}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o)$$

$$+f^{bo}_{1}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o)\Bigg\}+\mathcal{O}(d^{-3})\Bigg]$$

where $D_\theta = \frac{\partial}{\partial\theta}$ and $D_\phi \equiv \frac{\partial}{\partial\phi}$

We now express these scattered fields in terms of plane waves by

means of the following relations:

$$v(\theta_o,\phi_o) = e^{ik(x \sin\theta_o \cos\phi_o + y \sin\theta_o \sin\phi_o + z\cos\theta_o)},$$

$$v(\tfrac{\pi}{2},0) = e^{ikx} \quad , \quad v(\tfrac{\pi}{2},\pi) = e^{-ikx} \quad ,$$

$$v_{\theta_o}(\tfrac{\pi}{2},0) = -ikze^{ikx} \quad , \quad v_{\theta_o}(\tfrac{\pi}{2},\pi) = -ikze^{-ikx} \quad ,$$

$$v_{\phi_o}(\tfrac{\pi}{2},0) = ikye^{ikx} \quad , \quad v_{\phi_o}(\tfrac{\pi}{2},\pi) = -ikye^{-ikx} \quad ,$$

$$\frac{1}{2ik}\left[v_{\theta_o\theta_o}(\tfrac{\pi}{2},0) + v_{\phi_o\phi_o}(\tfrac{\pi}{2},0)\right] = \left[\frac{ik(y^2+z^2)}{2} - x\right] e^{ikx} \quad ,$$

$$\frac{1}{2ik}\left[v_{\theta_o\theta_o}(\tfrac{\pi}{2},\pi) + v_{\phi_o\phi_o}(\tfrac{\pi}{2},\pi)\right] = \left[\frac{ik(y^2+z^2)}{2} + x\right] e^{-ikx} \quad ,$$

Substitution of these relations into (3) and (4) yields:

$$(5) \quad U^o_{s_a} = e^{ikd}\Bigg[\frac{1}{d}v(\tfrac{\pi}{2},0)f_o^{ao}(\tfrac{\pi}{2},\theta_o,0,\phi_o) + \frac{1}{d^2}\Bigg(\frac{1}{2ik}\left[v_{\theta_o\theta_o}(\tfrac{\pi}{2},0)+v_{\phi_o\phi_o}(\tfrac{\pi}{2},0)\right]f_o^{ao}(\tfrac{\pi}{2},\theta_o,0,\phi_o)$$

$$+ v_{\theta_o}(\tfrac{\pi}{2},0)D_\theta f_o^{ao}(\tfrac{\pi}{2},\theta_o,0,\phi_o)$$

$$+ v_{\phi_o}(\tfrac{\pi}{2},0)D_\phi f_o^{ao}(\tfrac{\pi}{2},\theta_o,0,\phi_o)$$

$$+ v(\tfrac{\pi}{2},0)f_1^{ao}(\tfrac{\pi}{2},\theta_o,0,\phi_o)\Bigg) + \mathcal{O}(d^{-3})\Bigg],$$

and

$$(6)\quad U^o_{s_b} = e^{ikd}\left[\frac{1}{d}v(\tfrac{\pi}{2},\pi)f_o^{bo}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o) + \frac{1}{d^2}\left(\frac{1}{2ik}\left[v_{\theta_o\theta_o}(\tfrac{\pi}{2},\pi)+v_{\phi_o\phi_o}(\tfrac{\pi}{2},\pi)\right]f_o^{bo}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o)\right.\right.$$
$$+ v_{\theta_o}(\tfrac{\pi}{2},\pi)D_\theta f_o^{bo}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o)$$
$$+ v_{\phi_o}(\tfrac{\pi}{2},\pi)D_\phi f_o^{bo}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o)$$
$$\left.\left.+ v(\tfrac{\pi}{2},\pi)f_1^{bo}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o)\right) + \mathcal{O}(d^{-3})\right] .$$

These fields can then be expressed in terms of f_o alone by means of a recursion formula (7) obtained by Sommerfeld [14].

$$(7)\quad 2ik(n+1)f_{n+1} = \left\{n(n+1) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}f_n .$$

In our case $n = 0$, and the recursion reduces to

$$(8)\qquad 2ikf_1 = \left\{\cot\theta\, D_\theta + D_\theta^2 + \csc^2\theta\, D_\phi^2\right\} f_o \quad .$$

It is here that the proper choice of a coordinate system is seen to be convenient. We note that the recursion involves the Beltrami Operator which is singular at $\theta = 0$. This singularity occurs in the direction joining A to B, if the z axis is chosen collinear with d. Another difficulty in using such a coordinate system is the occurrence of terms in the expansions (3) and (4) which are not easily replaceable by derivatives of plane waves. The coordinate system we have selected overcomes these difficulties.

For $\theta = \frac{\pi}{2}$ and $\phi = 0$, (8) becomes

$$(9)\quad f_1(\tfrac{\pi}{2},\theta_o,0,\phi_o) = \frac{1}{2ik}\left\{D_\theta^2 + D_\phi^2\right\} f_o(\tfrac{\pi}{2},\theta_o,0,\phi_o)$$

For $\theta = \frac{\pi}{2}$ and $\phi = \pi$, (8) becomes

$$(10)\quad f_1(\tfrac{\pi}{2},\theta_o,\pi,\phi_o) = \frac{1}{2ik}\left\{D_\theta^2 + D_\phi^2\right\} f_o(\tfrac{\pi}{2},\theta_2,\pi,\phi_o)$$

Substitution of (9) and (10) into (5) and (6) results in expressions (11) and (12) for $U^o_{s_a}$ and $U^o_{s_b}$, which do not contain f_1^{ao} and f_1^{bo}.

$$\begin{aligned}(11)\quad U^o_{s_a} = e^{ikd}\Bigg[&\frac{1}{d}v(\tfrac{\pi}{2},0)f_o^{ao}(\tfrac{\pi}{2},\theta_o,0,\phi_o) + \frac{1}{d^2}\Bigg\{\frac{1}{2ik}\left[\left\{D^2_{\theta_o}+D^2_{\phi_o}\right\}v(\tfrac{\pi}{2},0)\right]f_o^{ao}(\tfrac{\pi}{2},\theta_o,0,\phi_o)\\
&+ D_{\theta_o}v(\tfrac{\pi}{2},0)D_\theta f_o^{ao}(\tfrac{\pi}{2},\theta_o,0,\phi_o)\\
&+ D_{\phi_o}v(\tfrac{\pi}{2},0)D_\phi f_o^{ao}(\tfrac{\pi}{2},\theta_o,0,\phi_o)\\
&+ \frac{1}{2ik}v(\tfrac{\pi}{2},0)\left\{D_\theta^2 + D_\phi^2\right\}f_o^{ao}(\tfrac{\pi}{2},\theta_o,0,\phi_o)\Bigg\}\\
&+\mathcal{O}(d^{-3})\Bigg]\end{aligned}$$

$$\begin{aligned}(12)\quad U^o_{s_b} = e^{ikd}\Bigg[&\frac{1}{d}v(\tfrac{\pi}{2},\pi)f_o^{bo}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o) + \frac{1}{d^2}\Bigg\{\frac{1}{2ik}\left[\left\{D^2_{\theta_o}+D^2_{\phi_o}\right\}v(\tfrac{\pi}{2},\pi)\right]f_o^{bo}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o)\\
&+ D_{\theta_o}v(\tfrac{\pi}{2},\pi)D_\theta f_o^{bo}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o)\\
&+ D_{\phi_o}v(\tfrac{\pi}{2},\pi)D_\phi f_o^{bo}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o)\\
&+ \frac{1}{2ik}v(\tfrac{\pi}{2},\pi)\left\{D_\theta^2+D_\phi^2\right\} f_o^{bo}(\tfrac{\pi}{2},\theta_o,\pi,\phi_o)\Bigg\}\\
&+ \mathcal{O}(d^{-3})\Bigg]\end{aligned}$$

3. Calculation of the perturbed far field amplitude.

The method employed previously in the two dimensional case will now yield, up to interaction terms of degree d^{-2}, the field scattered by the two bodies.

The final result must take into account the difference in phase of the original plane wave with respect to the two scatterers and also the difference in phase of the scattered fields of the two bodies at the point of observation. This can be done by a normalization of the coordinate systems and subsequent consideration of the phases in the normalized coordinate system. By a normalized coordinate system we mean a central coordinate system whose origin lies midway between the origins for A and B and whose x, y and z axes are parallel to those emanating from the origins of A and B. If we let α and β denote the angles of incidence corresponding to colatitude and longitude respectively (Figure 4), we obtain the following expression for the field scattered by the two bodies: (The subscript "o" is omitted from the f's for convenience.) (Here the incident wave is taken to have zero phase at the origin of the normalized system of coordinates.)

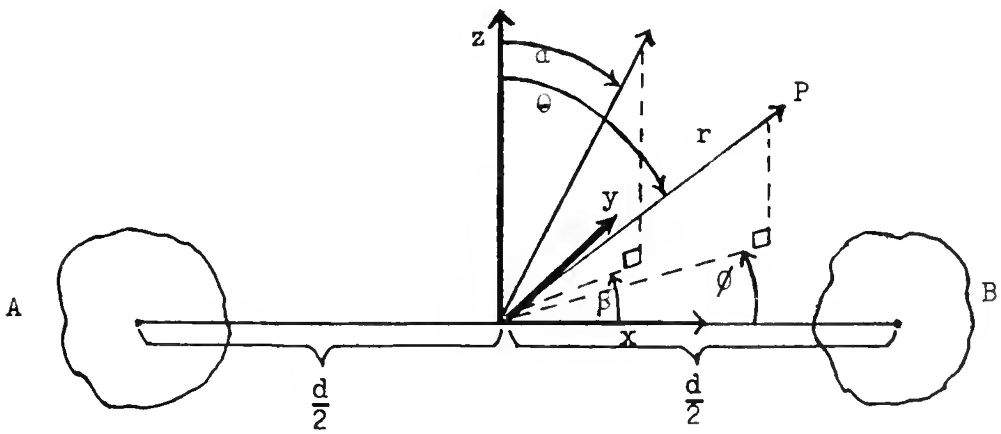

FIGURE 4

(13)

$$
\begin{aligned}
U_s \cong \frac{e^{ikr}}{r} \Bigg[\; & e^{ik\frac{d}{2}(\sin\theta\,\cos\phi - \sin\alpha\,\cos\beta)} f^{ao}(\theta,\alpha,\phi,\beta) + e^{-ik\frac{d}{2}(\sin\theta\,\cos\phi - \sin\alpha\,\cos\beta)} f^{bo}(\theta,\alpha,\phi,\beta) \\
& e^{ik\frac{d}{2}(\sin\theta\,\cos\phi - \sin\alpha\,\cos\beta)} \frac{e^{ikd}}{d} f^{ao}(\theta,\tfrac{\pi}{2},\phi,\pi) f^{bo}(\tfrac{\pi}{2},\alpha,\pi,\beta) + e^{-ik\frac{d}{2}(\sin\theta\,\cos\phi + \sin\alpha\,\cos\beta)} \frac{e^{ikd}}{d} f^{bo}(\theta,\tfrac{\pi}{2},\phi,0) f^{ao}(\tfrac{\pi}{2},\alpha,0,\beta) \\
& + e^{ik\frac{d}{2}(\sin\theta\,\cos\phi - \sin\alpha\,\cos\beta)} \frac{e^{ik2d}}{d^2} f^{ao}(\theta,\tfrac{\pi}{2},\phi,\pi) f^{bo}(\tfrac{\pi}{2},\tfrac{\pi}{2},\pi,0) f^{ao}(\tfrac{\pi}{2},\alpha,0,\beta) \\
& \qquad + e^{-ik\frac{d}{2}(\sin\theta\,\cos\phi - \sin\alpha\,\cos\beta)} \frac{e^{ik2d}}{d^2} f^{bo}(\theta,\tfrac{\pi}{2},\phi,0) f^{ao}(\tfrac{\pi}{2},\tfrac{\pi}{2},0,\pi) f^{bo}(\tfrac{\pi}{2},\alpha,\pi,\beta) \\
& + e^{ik\frac{d}{2}(\sin\theta\,\cos\phi + \sin\alpha\,\cos\beta)} \frac{e^{ikd}}{d^2} \Bigg\{ \frac{1}{2ik}\left[D^2_{\theta_o} + D^2_{\phi_o}\right] f^{ao}(\theta,\tfrac{\pi}{2},\phi,\pi) f^{bo}(\tfrac{\pi}{2},\alpha,\pi,\beta) + D_{\theta_o} f^{ao}(\theta,\tfrac{\pi}{2},\phi,\pi) D_\theta f^{bo}(\tfrac{\pi}{2},\alpha,\pi,\beta) \\
& \qquad + D_{\phi_o} f^{ao}(\theta,\tfrac{\pi}{2},\phi,\pi) D_\phi f^{bo}(\tfrac{\pi}{2},\alpha,\pi,\beta) + \frac{1}{2ik} f^{ao}(\theta,\tfrac{\pi}{2},\phi,\pi) \left[D^2_\theta + D^2_\phi\right] f^{bo}(\tfrac{\pi}{2},\alpha,\pi,\beta) \Bigg\} \\
& + e^{-ik\frac{d}{2}(\sin\theta\,\cos\phi + \sin\alpha\,\cos\beta)} \frac{e^{ikd}}{d^2} \Bigg\{ \frac{1}{2ik}\left[D^2_{\theta_o} + D^2_{\phi_o}\right] f^{bo}(\theta,\tfrac{\pi}{2},\phi,0) f^{ao}(\tfrac{\pi}{2},\alpha,0,\beta) + D_{\theta_o} f^{bo}(\theta,\tfrac{\pi}{2},\phi,0) D_\theta f^{ao}(\tfrac{\pi}{2},\alpha,0,\beta) \\
& \qquad + D_{\phi_o} f^{bo}_o(\theta,\tfrac{\pi}{2},\phi,0) D_\phi f^{ao}(\tfrac{\pi}{2},\alpha,0,\beta) + \frac{1}{2ik} f^{bo}(\theta,\tfrac{\pi}{2},\phi,0) \left[D^2_\theta + D^2_\phi\right] f^{ao}(\tfrac{\pi}{2},\alpha,0,\beta) \Bigg\} \\
& \qquad + \mathcal{O}(d^{-3}) \;\Bigg]
\end{aligned}
$$

4. An Explanation of the Result. The expression (13) for the scattered field of the combination of two bodies appears, at first glance, to be complicated. However, it is not as abstruse as it seems. A closer examination of this expression along the lines followed in the two dimensional case reveals the significance of the various terms and factors. We observe first that there are eight terms up to the order of accuracy d^{-2} calculated here, and we shall refer to them by number in the order of their appearance above. The first two terms represent single scattering while the others represent multiple scattering.

The functions f represent the complex scattering amplitude of the respective bodies with respect to an origin in the center of the relevant body, when the incident wave has zero phase at that center. The odd numbered terms represent the fields ultimately scattered by body A whereas the even numbered terms represent fields ultimately scattered by B.

The factors $e^{\pm ik\frac{d}{2}\sin\theta\cos\phi}$ represent the phase differences for the scatterers due to the use of the normalized coordinate system, while the factors $e^{\pm ik\frac{d}{2}\sin\alpha\cos\beta}$ take into account the phase of the incident wave at the center of the respective scatterers. The f's containing a θ and ϕ dependence are scattering patterns while the f's with specific values for θ and ϕ are excitation factors accumulated in the multiple scattering. We note, finally, that the factors of the form e^{iknd} where $n = 0,1,2$ refer to the increase ikd in the phase of a wave in going from one scatterer to another and that n signifies the number of bounces. If one uses the above comments as a guide, all the terms can be explained in a way similar to the explanation, given earlier, of the final expression for the scattered field in the two dimensional case.

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